New Methods To Generate Neutral Images For Spatial Pattern Recognition

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Abstract. Three new methods are developed to generate neutral spatial models for pattern recognition on raster data. The first method employs Genetic Programming (GP), the second Sequential Gaussian Simulation (SGS), and the third Conditional Pixel Swapping (CPS) in order to produce sets of "neutral images" that provide a probabilistic assessment of how unlikely an observed spatial pattern on a target image is under the null hypothesis. The sets of neutral images generated by the three methods are found to preserve different aspects of spatial autocorrelation on the target image. This preliminary research demonstrates the feasibility of using neutral image generation in spatial pattern recognition.

1 Introduction

Statistical Pattern Recognition is widely used in almost all spatial analysis software and appropriate null spatial models are an important component of tests for spatial pattern. Tests for spatial pattern proceed by calculating a statistic (e.g. spatial cluster statistic, autocorrelation statistic, etc) that quantifies a relevant aspect of an image's spatial pattern. The value of this statistic is then compared to the distribution of that statistic's value under a null spatial model. This provides a probabilistic assessment of how unlikely an observed spatial pattern is under the null hypothesis [9] [4].

But what should the null hypothesis be ? There is a long tradition of using complete spatial randomness (CSR) as the null spatial model, but this approach has been criticized as unrealistic since physical, biological, and social systems are rarely, if ever random.

Suppose two spatial systems, the first in which a spatial process (e.g. migration) is operating, the second in which the process is absent. Both systems include a common spatial process (diffusion) that operates regardless of the presence or absence of migration. The term "Neutral Model" captures the notion of a common, underlying process other than complete spatial randomness. Probabilistic pattern recognition then seeks to identify spatial patterns *above and beyond* that observed under the neutral model.

The identification of spatial patterns above and beyond that observed under the neutral model proceeds as follows: Starting from a given target image, the neutral signature is quantified. Using this signature, a set of neutral images is created by one of the methods described in this paper. Histograms with the distribution of the spatial pattern statistic of the neutral images can then be compared to the value of the spatial pattern statistic on the target image to provide a probabilistic assessment of how unlikely an observed spatial pattern is under the neutral model. This paper is concerned with identification and simulation of neutral spatial models that maintain aspects of the spatial structure of a target image.

1.1 Neutral Image Generation Methods

In principle there are many different methods possible to create neutral images. In this work we develop and experiment with three different methods that create a set of neutral images for a given target image of the same size: Genetic Programming, Sequential Gaussian Simulation, and Conditional Pixel Swapping.

Genetic Programming, in particular Symbolic regression, is used to evolve equations out of basic mathematical functions. These equations are then used to draw neutral images by supplying the pixel coordinates as input. The evolution of the equations proceeds by selection for certain spatial structures derived from the target images.

Sequential Gaussian Simulation (SGS) is another method for generating neutral images, which in this context are sometimes called realizations or simulations [3]. SGS is an algorithm for generating realizations of a multivariate Gaussian field. The grid locations are randomly picked for simulation and the values are influenced by neighboring, original data values (obtained by sampling at the site) and by neighboring, already-generated values. The simulation process is constrained by the distribution function describing the original data values in such a way that each simulation will conform to the original distribution function (i.e., the histogram of the original data and the simulated field will be very similar).

Each variable is simulated sequentially according to its normal ccdf (locally conditioned cummulative distribution function). The conditioning data consists of all original data and all previously simulated values found within a neighborhood of the location being simulated. The grid locations are randomly picked for simulation. The value generated for any grid location is influenced by neighboring, original data values (obtained by sampling at the site) and by neighboring, already-generated values. The simulation process is constrained by the distribution function describing the original data values in such a way that each simulation (total grid) will conform to the original distribution function (i.e., a histogram of the original data and a simulated field will be very similar). Because the CDF/PDF at all unsampled locations is influenced by its neighboring simulated values, the final simulation incorporates spatial continuity patterns inferred from the original data.

Conditional Pixel Swapping is based on the idea of simulated annealing. Neutral images are produced by initializing a random image with the same histogram as the target image. Each pixel of the image is then regarded as a discrete cell, and its value is swapped with another cell if this swapping improves the spatial structure as compared to the target.

1.2 Organization of the paper

This paper starts with a general description of the statistical criteria (e.g. spatial autocorrelation functions) which are used in order to create neutral images with the new methods. Then a detailed description of the three methods is given. Spatial pattern recognition statistics are examples of possible applications of neutral images and their targets. Two of those methods, in particular boundary detection methods, are described next. The paper concludes with experimental applications for all three methods and discussion of the results.

2 Statistical Criteria

The following statistical criteria are used for producing neutral images that preserve certain spatial aspects of the target.

2.1 Anselin's Local Moran

The local Moran test detects local spatial autocorrelation. It is related to Moran's I, a test for global spatial autocorrelation [5]. The local Moran decomposes Moran's I into contributions for each location, termed LISAs, for Local Indicators of Spatial Association. These indicators detect clusters of either similar or dissimilar values around a given observation.

The sum of LISAs for all observations is proportional to the global Moran's I. Thus, there can be two interpretations of LISA statistics, as indicators of local spatial clusters and as a diagnostic for outliers in global spatial patterns.
Anselin [1] defines a local Moran statistic for observation i:

$$I_i = (x_i - \bar{x}) \sum w_{ij}(x_j - \bar{x}) \tag{1}$$

where x_i and x_j are the x-values in area i or j and $\bar{x}$ is the mean value. w_{ij} is a weight denoting the strength of connection between areas i and j, developed from neighbor information. This weight ensures that only neighboring values of x_i are considered in the statistic, and weights are standardized to adjust for the number of neighbors.

The local Moran statistic I_i will be positive when values at neighboring locations are similar, and negative if they are dissimilar.

2.2 Global Moran's I

The global Moran's I is the sum of the LISAs of all pixels in the image. Since the LISAs can be positive or negative, depending on the spatial autocorrelation with the neighbors, the values can cancel each other out. Hence, some spatial information is lost when the global Moran's I is used.

2.3 Distribution Of Local Moran's I Values

One effect of using the global Moran's I in generation of neutral spatial images is that boundaries between correlated pixel values tend to be less sharp. The reason is that the images converge towards a global value, and the local structures are not considered. A way to deal with this problem is to keep track of the distribution of local Moran's I values. In other words, to construct a histogram for the whole image with bins for certain LISA values. When the algorithm is faithful to this histogram, the sharper boundaries in the target image are conserved. This approach has been tested in several applications in this research.

2.4 Bearing Correlogram

The Bearing Correlogram is a method for analyzing directional spatial autocorrelation that was developed by Rosenberg [7]. It is derived from the non directional measure of spatial autocorrelation, the local Moran's test. Test's like the local Moran are sometimes called one dimensional, as opposed to the Bearing Correlogram, which is considered two dimensional because it regards directionality in its statistics.

The Bearing Correlogram uses a non binary weight matrix, where the weight indicates not only the distance class involved but also the degree of alignment between the bearing of the two points and a fixed bearing.

The computation starts with the standard distance classes used in non directional correlograms. Each distance class has an associated weight matrix. For each distance class, each entry of its weight matrix, w_{ij}, is multiplied by the squared cosine of the angle between the points i and j, and a fixed bearing. If the original weights matrix was binary, this would not affect the 0 entries within the matrix, but down-weight the 1s based on their lack of association with the direction tested. I is calculated normally using the new weights matrix. In the current implementation no distance classes are used, and the values are only weighted by their contribution to the regarded angle. The procedure is repeated for a set of fixed bearings.

3 Null Model Algorithms

Three methods for the creation of neutral images are now described.

3.1 Genetic Programming

In its original form, Genetic Programming (GP) is the creation of computer programs by means of artificial evolution. The basic operation of GP is conceptually simple: (1) maintain a population of solutions to a problem, (2) select the better solutions for recombination with each other, and (3) use their offspring to replace poorer solutions. The combination of selection process and innovation generally leads to improved solutions, often the best found to date by any method. A good overview of the state of the art in this area is given by Banzhaf [2].

In this particular application the term Symbolic Regression is more appropriate. Symbolic Regression is a specialized form of Genetic Programming where the evolution is done on mathematical functions instead of whole computer programs. More specifically, in this application functions are used to draw neutral images are evolved.

The functions created by GP consist of general mathematical and trigonometric functions. The specific function set that was used is [-,*,/,log,sqrt,sin,cos,exp]. The evolved equations can be evaluated to create neutral images by taking the x and y coordinates of a two dimensional image as the input. Depending on those coordinates, the function will compute a color value for each individual pixel in the image.

The evolution of functions proceeds as follows: The initial population of equations is created with random combinations of the function sets described above. Then the equations are evaluated by computing the pixel values resulting from each equation for the neutral images and storing them in a two dimensional array of values. These values are then compared to the values of the target image by computing the sum of the squared differences for each neutral image. This sum depends on the statistical criteria that's evaluated in the current run. The neutral images are then ranked according to their fitness and a new generation is created by selecting the fittest (i.e. smallest errors) members of the population to produce offspring.

The evolution is continued for a preset number of generations. In this application it turned out to be of advantage to use a high number of individuals for each generation and a relatively low number of generations. Typical GP parameters were generation sizes of 1000 individuals and run times of 100 generations.

3.2 Sequential Gaussian Simulation

SGS is a straightforward algorithm for generating realizations of a multivariate Gaussian field. Consider the simulation of the continuous attribute z at N nodes $\mathbf{u}'_j$ of a grid (not necessarily regular) conditional to the data set $\{z(\mathbf{u}_\alpha), \alpha = 1, \ldots, n\}$. Sequential Gaussian simulation [8] proceeds as follows:

1. The first step is to check the appropriateness of the multiGaussian Random Function model, which calls for a prior transform of z-data into y-data with a standard normal cdf using the normal score transform.
2. The simulation is then performed in the normal space, that is on the y-data. The traditional implementation is the following:
   (a) Define a random path visiting each node of the grid only once.
   (b) At each node $\mathbf{u}_j$, determine the parameters (mean $m_Y(\mathbf{u}_j)$ and variance $\sigma_Y^2(\mathbf{u}_j)$) of the Gaussian conditional cumulative distribution function (ccdf) of the variable Y. This step requires solving a system of linear equations (kriging system) to compute the weights assigned to surrounding observations when deriving the ccdf mean (kriging estimate) and

variance (kriging variance):

$$m_Y(\mathbf{u}_j) = \sum_{\alpha=1}^{n} \lambda(\mathbf{u}_\alpha)\ y^{(l)}(\mathbf{u}_\alpha)$$
$$\sigma_Y^2(\mathbf{u}_j) = 1 - \sum_{\alpha=1}^{n} \lambda(\mathbf{u}_\alpha) \cdot C(\mathbf{u}_\alpha - \mathbf{u}_j)$$

(c) Sample randomly the ccdf and add the simulated value to the conditioning data set.
(d) Proceed to the next node along the random path, and repeat the two previous steps.
(e) Loop until all N nodes are simulated.

3. The final step consists of back-transforming the simulated normal scores $\{y^{(l)}(\mathbf{u}'_j),\ j = 1, \ldots, N\}$ into simulated values for the original variable, which amounts to applying the inverse of the normal score transform to the simulated y-values

Other realizations $\{y^{(l')}(\mathbf{u}_j), j = 1, \ldots, N\}$, $l' \neq l$, are obtained by repeating the procedure with a different random path.

Following a paper by Pebesma and Heuvelink (1999), the sequential simulation algorithm has been modified such that all L realization are generated during a single visit of the simulation grid. The new algorithm proceeds as follows:

1. Define a random path visiting each node of the grid only once.
2. At each node $\mathbf{u}_j$, solve the kriging system and use the kriging weights to compute, for each lth realization, the mean and variance of the ccdf as:

$$m_Y^{(l)}(\mathbf{u}_j) = \sum_{\alpha=1}^{n} \lambda(\mathbf{u}_\alpha)\ y^{(l)}(\mathbf{u}_\alpha)$$
$$\sigma_Y^2(\mathbf{u}_j) = 1 - \sum_{\alpha=1}^{n} \lambda(\mathbf{u}_\alpha) \cdot C(\mathbf{u}_\alpha - \mathbf{u}_j)$$

   Note that only the ccdf mean is realization-specific since the kriging variance does not depend on data values!
3. Sample randomly the L ccdfs and add the simulated values to the conditioning data set for each realization.
4. Proceed to the next node along the random path, and repeat the two previous steps.
5. Loop until all N nodes are simulated.

These two algorithms differ in the number of kriging systems that need to be solved. For the traditional implementation of SGS, a new random path is generated for each realization and the simulated values are then drawn from local normal ccdfs that have been constructed using both the sample data as well as previously simulated nodes. The previously simulated nodes, and hence the estimates for mean and variance at each node $\mathbf{u}_j$, are dependent on the realization. Thus, for each realization and each node a simple kriging system has to be solved, resulting in a total of N×L kriging systems. In contrast, the new implementation uses the same random path for all realizations and so the locations of data and previously simulated nodes to be taken into account for the construction of the local ccdfs are the same for all realizations. At each grid node it therefore suffices to determine the kriging weights once.

3.3 Conditional Pixel Swapping

Proposed by Jacquez, this approach produces neutral images by modifying incrementally a randomly generated image in a way which increases the similarity between it and the target image. We experimented with four measures of spatial auto-correlation with this method, the global Moran's I statistic and three other statistics based on Moran's I: the Moran correlogram, a directional correlogram which is sensitive to directional trends in an image, and the histogram of local Moran statistics. Each of these statistics is the sum of the local statistics associated with each pixel. This provided an efficient way of calculating the effect of modifying the values of individual pixels on the global statistic: If the value of pixel i is changed this will affect the local statistic at i, as well as the local statistics of neighboring pixels which are contingent on the value at i. If we calculated the difference between these local statistics (at i and its affected neighbors) before and after pixel i is changed, we knew the change of the image's global statistic as well. Thus our general approach was the following:

- Generate an initial neutral image at random. We accomplished this by setting each pixel of the neutral image to the value of randomly chosen pixels in the target. Other methods could be used to avoid sampling biases.
- Calculate the spatial autocorrelation statistic for the target and the initial neutral image.
- Swap two, randomly chosen pixels of the neutral image and calculate the effect on the neutral image's spatial statistic. If the new statistic is a better match to the target's the swap is kept. When histograms were used to represent the images' spatial autocorrelation swaps were kept if the sum of the squares of the differences of the bin heights was reduced.
- If the improvement to the the neutral image's statistic after the last swap is greater than some threshold, continue the swapping process. Otherwise, stop.

There were several ways to create neutral images from a given random image with the same histogram. One possibility was to visit all cells sequentially and change them according to the spatial structure desired. The disadvantage is that the spatial structure just created for each cell can be destroyed by changing the next cell in sequence. A way to limit this disadvantage is to visit all cells in random order. This way a change only takes place when it improves the spatial structure of the neutral image.

4 Pattern Recognition Statistics

The following two edge detection methods were applied on the target and neutral images as an application for spatial pattern recognition.

4.1 Wombling

Methods for delineating difference boundaries are called wombling techniques [10], and are calculated from Boundary Likelihood Values (BLV) to identify boundary elements. BLVs measure the spatial rate of change. Locations where variable values change rapidly are more likely to be part of a boundary; these locations have higher BLVs. The locations that have the highest BLV values are Boundary Elements (BEs) and are considered part of the boundary. Candidate BEs become part of the boundary when their BLVs exceed established thresholds. In crisp wombling, those BLVs with values above the threshold are assigned a Boundary Membership Value (BMV) of 1 (non-BEs have BMV = 0). In fuzzy wombling, BMVs can range between 0 and 1 and indicate partial membership in the boundary. The next step in delineating crisp difference boundaries is to connect BEs to create subboundaries. BEs are connected only if they are adjacent. With raster wombling, connection is based on the gradient angle of two adjacent BEs.

4.2 Gaussian Difference Boundaries

The second method that was used to detect significant edges in an image first transformed the pixel values into Gaussian space, then calculated the significant z-score difference for a specified alpha level, and finally examined all pairs of adjacent pixels to determine which had significant z-score differences.

In order to transform the pixel values it was necessary to sort and rank all the values in the image first. Each rank, r, could then be mapped to a z-score by dividing it by the number of ranks(pixels) and calculating the value of the inverse cumulative normal distribution function at $p = r/r_{total} - 0.5/r_{total}$.

Next, the distribution resulting from the difference of two random variables is $N[0, \sqrt{2}]$, so that a significant difference in z scores could be calculated by determining the z-score corresponding to a specified alpha level using the inverse c.d.f. described above and multiplying by $\sqrt{2}$.

Identifying the significant boundaries between pixels was simply a matter of comparing each difference in adjacent pixels' z-scores to the calculated cutoff. Finally, significant boundaries are connected to form a collection of boundaries with varying length and branchedness.

5 Prototype Applications

A Prototype application that employed the aforementioned methods was developed using C++ and a graphical user interface implemented with Microsoft

Foundation Classes. The user could load an example target image into the prototype and chose one of the three methods for the creation of neutral models. Once the neutral models were created the user could apply statistical pattern recognition as described above. An example application for each of the methods is discussed in following sections.

5.1 Genetic Programming

For small images GP was able to reproduce the target image with surprising accuracy. The accuracy depended almost exclusively on the GP parameters: The more time in generations, and the more individuals per generation were given to the GP, the more accurate reproduction of the target image was possible.

A first example of a target image was the 50x50 grey scale image shown in Figure 1 on the left. It was a Satellite Image of South Eastern Michigan showing the Detroit and Windsor metropolitan areas. The spatial criteria used here for the creation of neutral images for this example were the distribution of pixel values of the target image, which is the most stringent criteria. The best image that resulted from this criteria is shown in Figure 1 on the right.

Then neutral images were created using a Bearing Correlogram as the spatial structure to be preserved. A typical set of images created by the GP in this manner is shown in Figure 2.

5.2 Sequential Gaussian Simulation

An example for Sequential Gaussian Simulations is shown in Figure 3. The neutral image preserved the distribution and size of the lighter spots in the target image very well. SGS was able to produce a large number of neutral images in a short time, because once the kriging weights were computed only once and the same random path was used for each realization.

5.3 Conditional Pixel Swapping

The first example of CPS was a of 100x100 grey scale image taken from a 1 meter resolution satellite as shown in Figure 4 (top). The criteria used for the resulting set of neutral images was the bearing correlogram, using two bearings of zero and ninety degrees. A neighbourhood range of 4 rows of Moore's neighbourhood (full ring of neighbours) was used. The neutral images could reproduce the spatial structure of the target images (number and size of white spots) very well.

The range of neighbours included in the Bearing Correlogram has an influence on the spatial structure of the neutral images. In all applications the Moore neighbourhood was regarded. If more neighbours are included the neutral images turn out to have larger spots and the boundaries are not as sharp as with less neighbours. Consequently, the neutral images shown in Figure 4 have less sharper boundaries within the clusters. Furthermore, in this image 4 angles are regarded at the same time. Because of this, the directionality of the neutral images is not so distinctive as in the following images.

As a second example, the algorithm was used to create neutral images from the simple Global Moran's I of the target image shown in Figure 5 (top). Again, the neutral images show a distribution of dark and white spots, and clustering that is very similar to the target.

The same target image was used for the next test shown in Figure 6. This time however, the neutral images were produced on basis of the Bearing Correllogram. The Bearing for this Bearing Correlogram was taken to be the angle of the strongest directionality in the target image. The resulting neutral images show a strong directionality from the lower left to the upper right of the picture. The clustering and distribution of dark and bright pixel values however is again very close to the target image.

The following set of figures shows how CPS was used to produce neutral images for a larger target, which in turn were used for statistical pattern recognition with edge detection. Figure 7 shows the target image displayed in the prototype application. The Wombling method has been applied to this image and the edges are shown on top of the original image. Wombling has also been applied to a set of 50 neutral images, one of which is shown in Figure 8. Again, the edges are displayed on top of the neutral images. Figure 9 finally shows a histogram of the detected sub boundary lengths for the neutral images compared to the sub boundary lengths of the target image. It is obvious that the sub boundary lengths of the target image are significantly longer than those of the neutral images. This implies that the target image has an underlying process that is more complex or different from the spatial structure created under the neutral hypothesis.

Finally, there is an example that demonstrates the application of neutral models at a very small scale. The target image is a microscopic picture of two filament systems taken from a body cell. This image is shown together with a CPS neutral image in Figure 10. The neutral image was produced with bearings corresponding to the directionality of the original image.

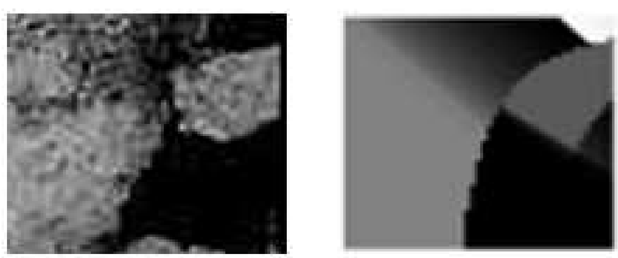

Fig. 1. Example Target Image (left) and Neutral Image (right) created with Genetic Programming. The Target is a Satellite Image of South Eastern Michigan showing the Detroit and Windsor metropolitan areas

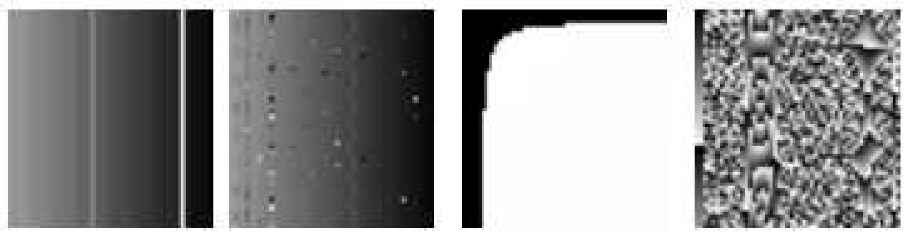

Fig. 2. Images created from Bearing Corellogram with Genetic Programming

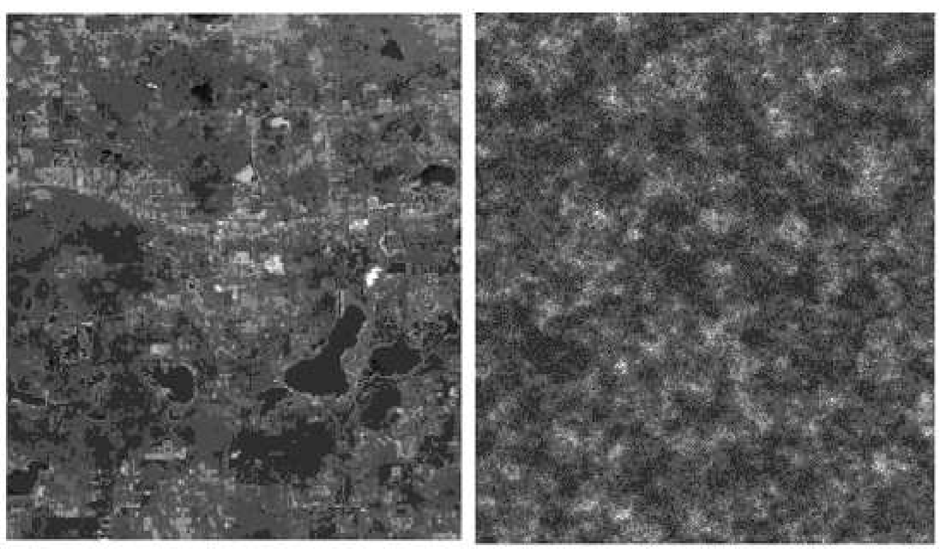

Fig. 3. Target (left) and Neutral Image created with Gaussian Sequential Simulation

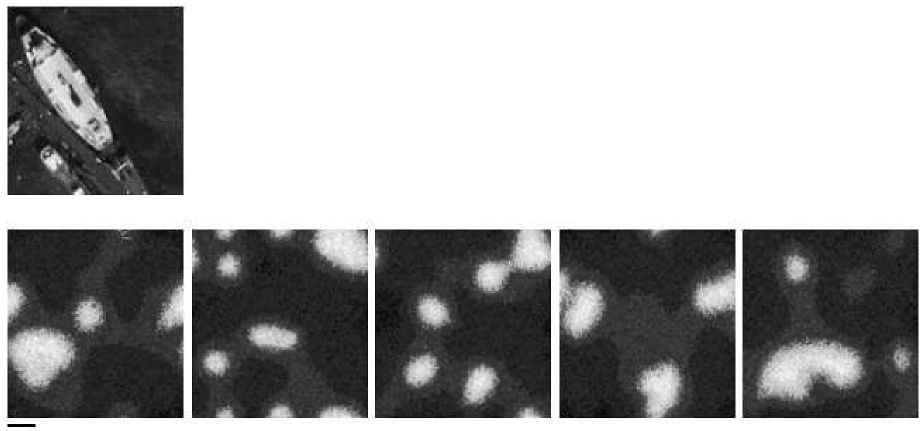

Fig. 4. CPS Neutral images created from Bearing Correlogram with a range of 4 neighbors

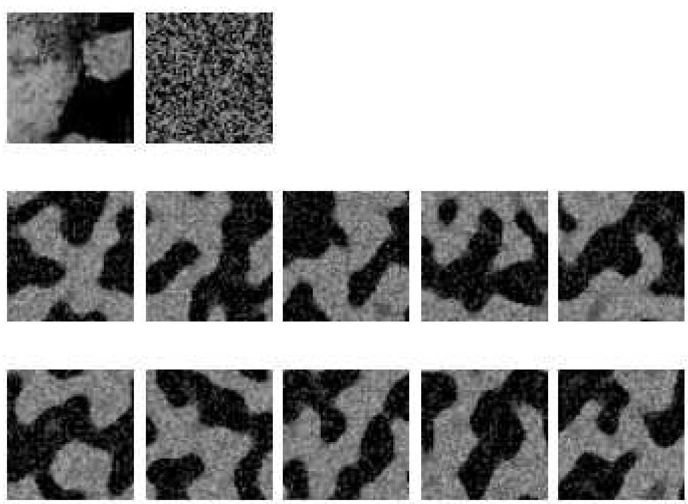

Fig. 5. CPS Neutral Images created from conservation of global Moran's I of target. On top target image and image resulting from random sampling from histogram

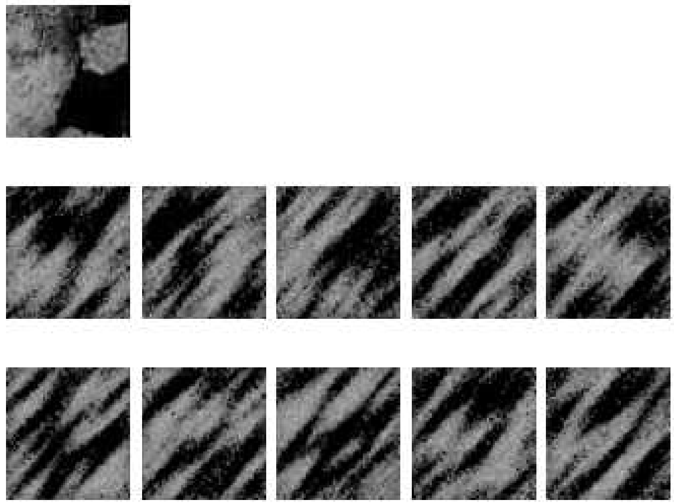

Fig. 6. CPS Neutral images created from Bearing Correlogram, using only the angle of strongest directionality

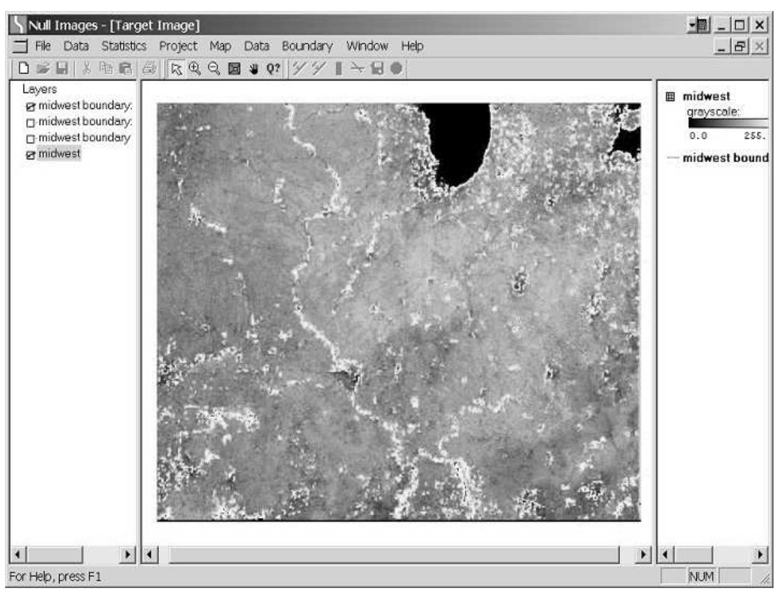

Fig. 7. Edgedetection on target image

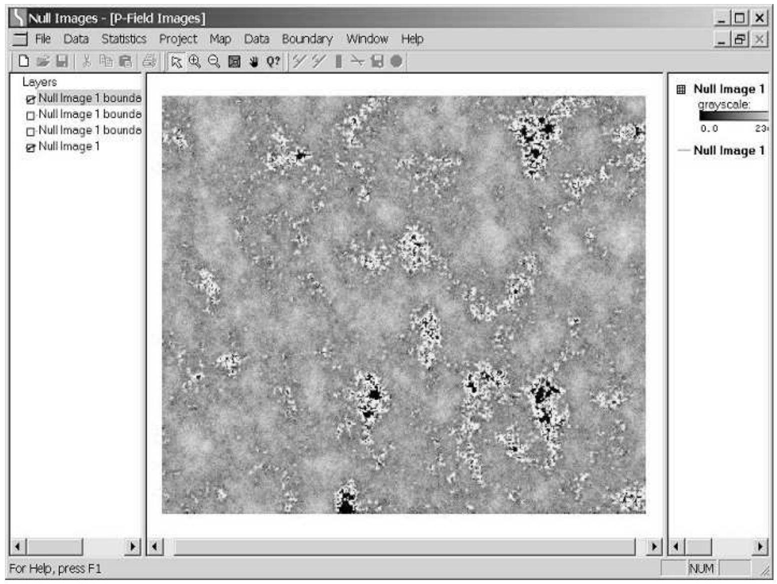

Fig. 8. Edgedetection on neutral image

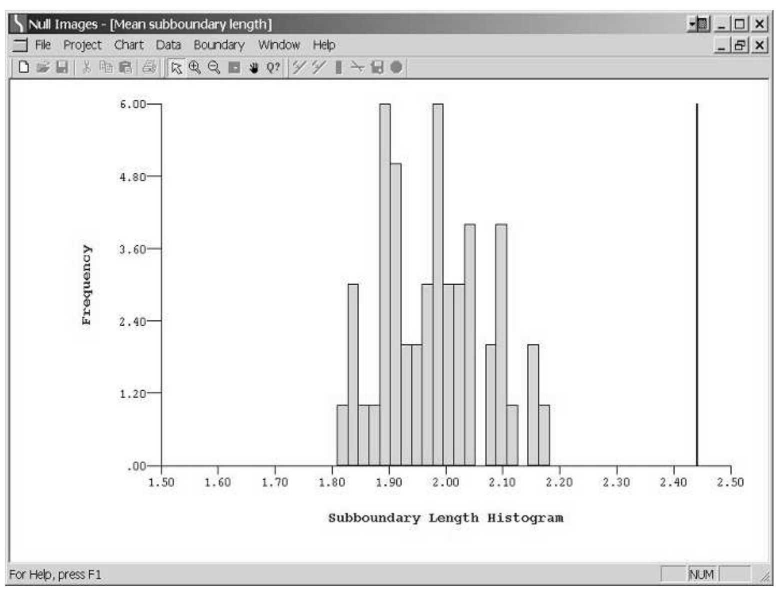

Fig. 9. Comparison on sub boundary lengths on target and neutral images

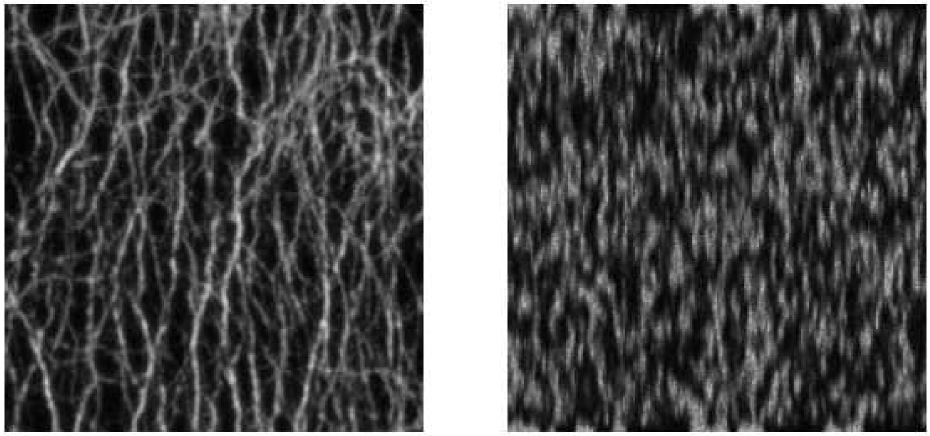

Fig. 10. CPS Filaments Example with a range of 2 neighbours, Bearing Correlogram, angle 90 degrees

6 Summary And Conclusions

Three methods were developed that produced sets of neutral images that preserved certain spatial structures on a target image such as local autocorrelation, global autocorrelation, or directional autocorrelation.

Each of the methods had different advantages and disadvantages in regard to the quality of the neutral images and the speed of their creation. Genetic Programming had the advantage of producing very diverse neutral images that differed greatly from the original image but still conserved the required spatial property. The disadvantage was that it would take a long time for them to evolve, especially for larger target images. Sequential Gaussian Simulation had the advantage of being able to create large images quickly. It also conserved the required spatial properties and was able to work with different spectral bands and on different areas of the target image. The disadvantage of SGS was that it was difficult to automatically fit an appropriate variogram model to the target image. Conditional Pixel Swapping method was very flexible and could also work on large images. The resulting neutral images looked close to the original image. Due to its relative simplicity, speed, and good results, this seemed to be a promising method.

As a test for spatial pattern recognition statistics, two methods were devised for edge detection: Wombling and Gaussian Differencing. They could easily detect interesting features in the target images and support those quantitatively by comparison with the neutral images.

Why should one use a neutral model that incorporates some spatial autocorrelation as opposed to CSR ? In real world systems, complete spatial randomness is an extraordinary exception that is rarely observed. Employing CSR as a null hypothesis in spatial pattern recognition poses a "strawman" that will almost certainly be rejected. The scientific value of such a simplistic null hypothesis is therefore quite small, since rejecting a simplistic null hypothesis has little inferential yield. The neutral models we have presented capture spatial autocorrelation signatures representative of real world systems. They make it possible to test for spatial patterns *above and beyond* that observed in the absence of an alternative spatial process (e.g. boundary formation). This strengthened inference structure is expected to lead to scientific insights that are not possible using untenable and unreasonable null hypothesis such as CSR.

This preliminary research demonstrated the feasibility of using neutral image generation in statistical pattern recognition. Our future research will focus on the application of these techniques to large multivariate spatial fields arising from high resolution hyper spectral images.

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